Exercise 2.3.1

(Exact solution of logistic equation) There are two ways to solve the logistic equation $\dot{N} = rN(1 - N/K)$ analytically for an arbitrary initial condition N_0.

a) Separate variables and integrate, using partial fractions.

b) Make the change of variables $x = 1/N$. Then derive and solve the resulting differential equation for x.

Solution

Part (a)

Solve the logistic equation by separating variables and integrating both sides.

$$\frac{dN}{dt} = \frac{r}{K}N(K - N)$$

$$\frac{dN}{N(K-N)} = \frac{r}{K}\,dt$$

$$\int \frac{dN}{N(K-N)} = \int \frac{r}{K}\,dt$$

$$\int \left(\frac{\frac{1}{K}}{N} - \frac{\frac{1}{K}}{N-K}\right) dN = \int \frac{r}{K}\,dt$$

$$\frac{1}{K}\ln|N| - \frac{1}{K}\ln|N-K| = \frac{r}{K}t + C$$

$$\ln|N| - \ln|N-K| = rt + CK$$

$$\ln\left|\frac{N}{N-K}\right| = rt + CK$$

$$\left|\frac{N}{N-K}\right| = e^{rt+CK}$$

Place $\pm$ on the right side in order to remove the absolute value sign.

$$\frac{N}{N-K} = \pm e^{CK}e^{rt}$$

Use a new constant A for $\pm e^{CK}$.

$$\frac{N}{N-K} = Ae^{rt} \tag{1}$$

Now apply the initial condition $N(0) = N_0$ to determine A.

$$\frac{N_0}{N_0 - K} = A$$

As a result, equation (1) becomes

$$\frac{N}{N-K} = \frac{N_0}{N_0-K}e^{rt}$$

$$N = N\frac{N_0}{N_0-K}e^{rt} - K\frac{N_0}{N_0-K}e^{rt}$$

$$\left(1 - \frac{N_0}{N_0-K}e^{rt}\right)N = -K\frac{N_0}{N_0-K}e^{rt}$$

$$\frac{N_0 - K - N_0e^{rt}}{N_0-K}N = -\frac{KN_0e^{rt}}{N_0-K}.$$

Multiply both sides by $N_0 - K$.

$$[N_0(1-e^{rt}) - K]N = -KN_0e^{rt}$$

Therefore,

$$N(t) = \frac{KN_0e^{rt}}{K - N_0(1-e^{rt})}.$$

Part (b)

The same initial value problem will be solved here but with a different method.

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right), \quad N(0) = N_0$$

Make the change of variables,

$$x = \frac{1}{N},$$

in the ODE and the initial condition.

$$\frac{d}{dt}\left(\frac{1}{x}\right) = \frac{r}{x}\left(1 - \frac{1}{Kx}\right), \quad x(0) = \frac{1}{N(0)} = \frac{1}{N_0}$$

Simplify the transformed ODE.

$$\left(-\frac{1}{x^2}\right)\frac{dx}{dt} = \frac{r}{x}\left(1 - \frac{1}{Kx}\right)$$

$$\frac{dx}{dt} = r\left(\frac{1}{K} - x\right)$$

$$\frac{dx}{dt} + rx = \frac{r}{K} \tag{2}$$

This is a first-order linear ODE, so it can be solved with an integrating factor I.

$$I = \exp\left(\int^t r\,ds\right) = e^{rt}$$

Multiply both sides of equation (2) by I.

$$e^{rt}\frac{dx}{dt} + re^{rt}x = \frac{r}{K}e^{rt}$$

Rewrite the left side as a derivative using the product rule.

$$\frac{d}{dt}(e^{rt}x) = \frac{r}{K}e^{rt}$$

Integrate both sides with respect to t.

$$e^{rt}x = \frac{1}{K}e^{rt} + D \tag{3}$$

Apply the initial condition now to determine D.

$$\frac{1}{N_0} = \frac{1}{K} + D \quad \to \quad D = \frac{1}{N_0} - \frac{1}{K} = \frac{K - N_0}{KN_0}$$

Consequently, equation (3) becomes

$$e^{rt}x = \frac{1}{K}e^{rt} + \frac{K - N_0}{KN_0}.$$

Solve for x.

$$\begin{aligned}
x(t) &= \frac{1}{K} + \frac{K - N_0}{KN_0e^{rt}} \\
&= \frac{N_0e^{rt}}{KN_0e^{rt}} + \frac{K - N_0}{KN_0e^{rt}} \\
&= \frac{N_0e^{rt} + K - N_0}{KN_0e^{rt}} \\
&= \frac{K + N_0(e^{rt} - 1)}{KN_0e^{rt}} \\
&= \frac{K - N_0(1 - e^{rt})}{KN_0e^{rt}}
\end{aligned}$$

Therefore, since $N(t) = 1/x(t)$,

$$N(t) = \frac{KN_0e^{rt}}{K - N_0(1 - e^{rt})}.$$